Transient growth in networked systems

(LAPTOP REQUIRED)

Non-normality is a very active topic of research, in particular due to the growing interest for network science. In this exercise, you will reproduce the results of the recent publication [1].

Complex interactions can often be encapsulated in a graph representation through the so-called *Laplacian* matrix $\mathbf{L}$, whose entries $[\mathbf{L}]_{ij} = 1$ if and only if there is an edge **from** j **to** i (which does *not* imply the reciprocal relation, the direction is important). If there is also an edge from i to j determines whether the network is directed or not. We will see that this condition can have a strong impact on the system dynamics. Furthermore, for diagonal terms $[\mathbf{L}]_{ii}$, one must subtract the number of edges **leaving** the node i. As an example, let us consider the following linear (presumably linearized) dynamical system made of N coupled components:

$$\frac{d\mathbf{x}}{dt} = \mathbf{M}\mathbf{x}, \quad \text{with} \quad \mathbf{M} = -a\mathbf{I} + D\mathbf{L} \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is the vector of the system states. The dynamics described by the system (1) is twofold : (i) each state x_i experiences the same internal dissipation with a decay rate of a, as embedded in the term "$-a\mathbf{I}$" ; (ii) if an edge exists connecting a node i to a node j, the state x_i transforms into x_j at a rate D ; this is encoded in the so-called "diffusion" term "$D\mathbf{L}$". The ratio D/a thus compares the "diffusion" with respect to the dissipation.

To make it clear, let us consider for instance the network shown in Fig. 1

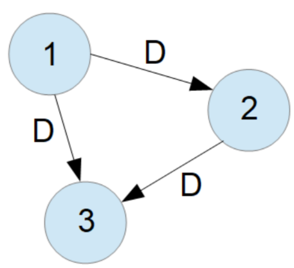

Figure 1: A example of directed network with 3 components.

This network corresponds to

$$\mathbf{L} = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \tag{2}$$

in turns corresponding to the system

$$\begin{aligned}\dot{x}_1 &= -ax_1 - 2Dx_1 \\ \dot{x}_2 &= -ax_2 + D(x_1 - x_2) \\ \dot{x}_3 &= -ax_3 + D(x_1 + x_2).\end{aligned} \tag{3}$$

Q1) Consider now the networks schemed in Fig. 2, both constituted of eleven nodes.

Figure 2: (a) A network with a full rotational symmetry, meaning each vertex has one incoming link and an outgoing one. It is constituted of $N = 11$ elements.(b) The same network but an edge (at a random place) has been removed.

The network in Fig. 2a has a full rotational symmetry, meaning that each vertex has one incoming link and an outgoing one. By choosing $a = 0.1$ and $D = 10$ (implying $D/a \ll 1$) Compute the matrix $\mathbf{M}$ associated to the network in Fig. 2a and answer the following question

(a) Is $\mathbf{M}$ normal or non-normal ?

(b) Compute the eigenvalues and eigenvectors of $\mathbf{M}$; is the system stable or unstable ?

(c) For a range of temporal horizons $T > 0$, compute the optimal gain $G(T)$ of the system, whose definition is recalled in (4), and the associated optimal initial conditions.

$$G(T) = \max_{\mathbf{x}(0)} \frac{||\mathbf{x}(T)||}{||\mathbf{x}(0)||} = \max_{\mathbf{x}(0)} \frac{||e^{\mathbf{M}T}\mathbf{x}(0)||}{||\mathbf{x}(0)||} = ||e^{\mathbf{M}T}|| \tag{4}$$

Plot $G(T)$ as a function of T : do you observe some transient growth ?

(d) Compares $G(T)$ with $e^{\Re(\sigma_1)T}$, where σ_1 is the least stable/most unstable eigenvalue (and $\Re(\cdot)$ the real part) ; also compare the optimal initial conditions with the eigenvector associated with this eigenvalue : what do you observe ? Comment.

Q2) Demonstrate the observations made in Q1d) analytically. You may admit that if a matrix is normal, its eigenvectors form an orthonormal basis. You may also admit that if $\mathbf{M} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}$ where $\mathbf{P}$ is the matrix of eigenvector of $\mathbf{M}$ and $\boldsymbol{\Lambda}$ the diagonal matrix of its eigenvalues, then $e^{\mathbf{M}t} = \mathbf{P}e^{\boldsymbol{\Lambda}t}\mathbf{P}^{-1}$ (where taking the matrix exponential of a diagonal matrix amounts to taking the exponential of the term on its diagonal).

Q3) We consider now the network in Fig 2b, which is the same as the one in Fig 2a, at the exception that an edge has been removed (at a random place).

1. Repeat Q1) entirely.
2. Plot the evolution of $\mathbf{x}(t) = e^{\mathbf{M}t}\mathbf{x}_{\text{opt}}(0)$ as a function of time t, where $\mathbf{x}_{\text{opt}}(0)$ is the optimal initial condition associated with the optimal temporal horizon (leading the the largest gain over all possible T) : what do you observe ? Can you propose a physical interpretation for the transient growth occurring in this network ? Why didn't it occur with the previous network ?
3. Can you make a parallel with transient growth mechanisms in shear flows seen in class ? For instance the lift-up mechanism ?
4. Investigate the influence of the ratio D/a on your results : what do you observe ? Can you explain it physically ?

Q4) In another recent publication [2], a collection of empirical networks and some of their properties is reported in Table 2 (of the paper) : what is the striking fact concerning the non-normality of all of these networks ? Can you think of consequences of non-normality in real life, for instance in the propagation of epidemics ?

Q5) Nevertheless, it is important to understand that the degree of non-normality is relative to the choice of a norm (which may or may not be evident). Consider now a norm given by a weight matrix $\mathbf{B}$, defined as $||\mathbf{x}(T)||_{\mathbf{B}} = \mathbf{x}(T)^H\mathbf{B}\mathbf{x}(T)$. We seek the optimal transient gain under this new norm :

$$G(T) = \max_{\mathbf{x}(0)} \frac{||\mathbf{x}(T)||_{\mathbf{B}}}{||\mathbf{x}(0)||_{\mathbf{B}}} \tag{5}$$

(a) If we choose for instance $\mathbf{B} = (\mathbf{P}^{-1})^H\mathbf{P}^{-1}$ (H the Hermitian transpose), demonstrate analytically that the optimal gain is $G(T) = e^{\Re(\sigma_1)T}$ regardless of $\mathbf{M}$ (as long as it is diagonalizable).

(b) In the case of Navier-Stokes equations, which norm do we usually seek to optimize ? Why ?

References

[1] Asllani, M., and Carletti, T. Topological resilience in non-normal networked systems. *Phys. Rev. E 97* (Apr 2018), 042302.

[2] Asllani, M., Lambiotte, R., and Carletti, T. Structure and dynamical behavior of non-normal networks. *Science Advances 4*, 12 (2018), eaau9403.